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Polyconvexity of generalized polynomial-type hyperelastic strain energy functions for near-incompressibility

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Abstract

In this article we investigate several models contained in the literature in the case of near-incompressibility based on invariants in terms of polyconvexity and coerciveness inequality, which are sufficient to guarantee the existence of a solution. These models are due to Rivlin and Saunders, namely the generalized polynomial-type elasticity, and Arruda and Boyce. The extension to near-incompressibility is usually carried out by an additive decomposition of the strain energy into a volume-changing and a volume-preserving part, where the volume-changing part depends on the determinant of the deformation gradient and the volume-preserving part on the invariants of the unimodular right Cauchy–Green tensor. It will be shown that the Arruda–Boyce model satisfies the polyconvexity condition, whereas the polynomial-type elasticity does not. Therefore, we propose a new class of strain-energy functions depending on invariants. Moreover, we focus our attention on the structure of further isotropic strain-energy functions.

Keywords: Hyperelasticity; Near-incompressibility; Polyconvexity; Parameter identification; Existence theorems

1. Introduction

Most solid materials show in the finite strain range nearly incompressible, i.e. weakly compressible material behaviour. The constitutive models concerned usually utilize hyperelasticity relations which describe one part of the model. This is done in the case of rate-dependent and rate-independent constitutive theories such as viscoelasticity, elastoplasticity or viscoplasticity. Naturally, the investigations below are also of interest in the case of purely elastic material behaviour. In analytical derivations the weakly compressible behaviour, which can be observed in most experiments, is replaced by the assumption of incompressibility in order to obtain particular solutions. On the other hand, it is much more convenient in the numerical treatment of these constitutive models, for example, utilizing the finite element method, to employ the nearly incompressible extension. In this case there are three usually employed constitutive

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models in use, namely the generalized polynomial-type elasticity due to Rivlin and Saunders, the Arruda and Boyce as well as Ogden's model (Rivlin and Saunders, 1951; Arruda and Boyce, 1993; Ogden, 1972a). The strain-energy functions concerned are originally formulated in the case of incompressibility as functions of the first and second invariant or in eigenvalues of the left (or right) Cauchy–Green tensor. One way of extending these models to the nearly incompressible case is to exchange the invariants or eigenvalues of the original Cauchy–Green tensors, i.e. in the modified model one uses the unimodular part of the Cauchy–Green tensors which are based on the multiplicative decomposition of the deformation gradient into a volume-changing and a volume-preserving part. This decomposition goes back to Flory (1961). Furthermore, additional parts of the strain-energy function depend merely on the determinant of the deformation gradient, i.e. the strain energy decomposes additively into two parts: one part depends on the volume-changing part via the determinant of the deformation gradient and the other part on the invariants or eigenvalues of the right Cauchy–Green tensor built up by the volume-preserving part of the deformation gradient. This article discusses this modification of the models with respect to physical and mathematical aspects.

A much debated question in the area of finite elasticity is the correct formulation of constitutive inequalities ensuring reasonable solutions to physical problems. Since we are focusing on hyperelastic material behaviour these constitutive inequalities translate into conditions on the free energy. The so-called Baker–Ericksen inequalities, the Coleman–Noll condition or Hill's inequality may serve as examples for the earlier attempts to establish these correct formulations. For a discussion on these inequalities see, for example, Baker and Ericksen (1954), Marsden and Hughes (1983), Truesdell and Noll (1965), Wang and Truesdell (1973), Hill (1970) and Ogden (1984). However, these inequalities could not be proved to guarantee the well-posedness of the problem. Moreover, some criteria cannot be shown to be satisfied a priori.

The mathematical treatment of the corresponding boundary-value problem (the structural mechanics problem) is mainly based on the direct methods of variation, i.e. to find a minimizing deformation of the elastic free energy subject to the specific boundary conditions. This minimizing deformation is found by constructing infimizing sequences of deformations and then showing that the sequence converges in some sense to the sought minimizer. The main ingredient for carrying out this program is a quasi-convexity hypothesis on the free energy, Morrey (1952): roughly, it ensures that the functional to be minimized is weakly lower semi-continuous. However, this condition is difficult to handle since it is a non-local integral condition. A much more tractable condition has been introduced by Ball in his seminal article (Ball, 1977a), it is the so-called polyconvexity condition. There exists a vast literature on polyconvexity (see e.g., Ball, 1977a,b; Marsden and Hughes, 1983; Ciarlet, 1988; Charrier et al., 1988 and the literature cited there) and fortunately some energy expressions already introduced are covered by this concept (Ogden's, Mooney-Rivlin and Neo-Hookean model). For isochoric-volumetric decouplings some forms of polyconvex energies have been proposed by Charrier et al. (1988) or Dacorogna (1989, p. 134 and pp. 256ff.). There are some simple stored energies of St. Venant-Kirchhoff type (Ciarlet, 1988; Raoult, 1986) or energies involving the so-called (logarithmic) Hencky tensor which, however, do not satisfy the polyconvexity condition (Neff, 2000). Moreover, it can be shown that neither St.Venant-Kirchhoff nor strain-energy functions based on Hencky strains lead to elliptic equilibrium conditions.

It can be shown that polyconvexity of the stored energy implies that the corresponding acoustic tensor is elliptic for all deformations whatsoever, moreover strict ellipticity is sufficient for the Baker–Ericksen inequalities to hold. The precise difference between the local property of ellipticity and quasi-convexity is still an active field of research, since counterexamples exist which are elliptic throughout but not quasi-convex. However, these examples are neither frame indifferent nor isotropic; from a purely mechanical point of view this difference might be negligible. Polyconvexity as such does not conflict with the possible non-uniqueness of equilibrium solutions, since it only guarantees the existence of at least one minimizing deformation. It is possible that several metastable states (local equilibria) and several absolute minimizers exist, even so, one might conjecture that apart from trivial symmetries the absolute minimizer is unique, at least for the pure Dirichlet boundary-value problem. In general, under polyconvexity assumptions, no claim can be made as to the stability or smoothness of the solution, apart from the natural statement that the minimizer lies in the Sobolev space considered. Moreover, it is not known that the minimizing deformation is a weak solution of the local balance equation, due to possible singularities in the deformation gradient. We remark, following (Ball, 1977a, p. 398) that polyconvexity implies the existence for all boundary conditions and body forces which might be somewhat unrealistic. The conclusion that one particular form of the stored energy is not polyconvex does not mean that this energy should be ruled out from the outset. Indeed, the corresponding failure of ellipticity at large deformation gradients only (see the examples below) might be physically correct, indicating, for example, the onset of fracture or some other local instability like the formation of microstructure. On the other hand, the proof that some energy is elliptic for a reasonable range of deformation is presently not enough to establish an existence theorem. Since we will be concerned with nearly incompressible isotropic hyperelasticity only where we expect neither fracture nor microstructure, the polyconvexity assumption seems to be a convenient mathematical tool to ascertain the existence of a minimizer of an elastic free energy.

Our main contribution consists of enlarging the class of known polyconvex energies including expressions which bear resemblance to the generalized polynomial-type elasticity relations due to Rivlin and Saunders (1951), defined by modified invariants mentioned above. To our knowledge general polynomial energy expressions like those of Rivlin and Saunders (1951) or Arruda and Boyce (1993) have not been investigated with respect to polyconvexity. The polyconvexity conditions will then translate into a requirement on the structure of the polynomial terms and the restriction to positive material parameters.

2. Strain-energy functions

We start with the multiplicative decomposition of the deformation gradient $\mathbf{F} = \operatorname{Grad}\vec{\phi}(\vec{X}, t)$ of a material point $\vec{X}$ at time t into a volume-changing and a volume-preserving part

$$\mathbf{F} = \widehat{\mathbf{F}}\,\overline{\mathbf{F}}. \tag{2.1}$$

$\vec{x} = \vec{\phi}(\vec{X}, t)$ denotes the deformation. Bold-face roman letters denote tensorial quantities. The volume-preserving part

$$\overline{\mathbf{F}} = J^{-1/3}\mathbf{F}, \qquad \det\overline{\mathbf{F}} = 1, \tag{2.2}$$

$J \equiv \det\mathbf{F}$, and the volume-changing part $\widehat{\mathbf{F}} = J^{1/3}\mathbf{I}$, are used to define unimodular left and right Cauchy–Green tensors

$$\overline{\mathbf{C}} = \overline{\mathbf{F}}^{\mathrm{T}}\overline{\mathbf{F}}, \qquad \overline{\mathbf{B}} = \overline{\mathbf{F}}\,\overline{\mathbf{F}}^{\mathrm{T}}, \tag{2.3}$$

$\det\overline{\mathbf{C}} = \det\overline{\mathbf{B}} = 1$, which can be expressed by the original Cauchy–Green tensors $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ via

$$\overline{\mathbf{C}} = J^{-2/3}\mathbf{C} \quad \text{and} \quad \overline{\mathbf{B}} = J^{-2/3}\mathbf{B}. \tag{2.4}$$

Now, we define the two sets of invariants

$$\mathrm{I_C} = \operatorname{tr}\mathbf{C}, \quad \mathrm{II_C} = \tfrac{1}{2}\left((\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\mathbf{C}^2\right), \quad \mathrm{III_B} = \det\mathbf{C} \tag{2.5}$$

and

$$\mathrm{I}_{\overline{\mathbf{C}}} = \operatorname{tr}\overline{\mathbf{C}}, \quad \mathrm{II}_{\overline{\mathbf{C}}} = \tfrac{1}{2}\left((\operatorname{tr}\overline{\mathbf{C}})^2 - \operatorname{tr}\overline{\mathbf{C}}^2\right) = \operatorname{tr}\overline{\mathbf{C}}^{-1} = \operatorname{tr}\operatorname{adj}\overline{\mathbf{C}}. \tag{2.6}$$

Eq. $(2.6)_2$ becomes obvious in view of the evaluation of condition $\det \overline{\mathbf{C}} = 1$ and Cayley-Hamilton's theorem, see Eq. (A.1) of Appendix A. $\operatorname{adj} \mathbf{A} = (\det \mathbf{A})\mathbf{A}^{-1}$ denotes the adjugate and $\operatorname{tr} \mathbf{A} = a_{ii}$ defines the trace of a second order tensor.

In order to obtain a basic representation with respect to the polyconvexity condition, we must express the invariants by the deformation gradient $\mathbf{F}$, by the determinant J and by the adjugate $\operatorname{adj} \mathbf{F} = (\det \mathbf{F})\mathbf{F}^{-1}$:

$$\mathrm{I}_{\overline{\mathrm{B}}}(J, \mathbf{F}) = \frac{\|\mathbf{F}\|^2}{(\det \mathbf{F})^{2/3}}, \qquad \mathrm{II}_{\overline{\mathrm{B}}}(J, \operatorname{adj} \mathbf{F}) = \frac{\|\operatorname{adj} \mathbf{F}\|^2}{(\det \mathbf{F})^{4/3}}, \tag{2.7}$$

$\|\mathbf{F}\|^2$ defines the scalar product of the deformation gradient, $\|\mathbf{F}\|^2 = \langle \mathbf{F}, \mathbf{F} \rangle$.

In the following, we study some frequently used strain-energy functions in the cases of incompressibility and their extension to near-incompressibility in view of the existence of the solutions (polyconvexity and coerciveness). Furthermore, we develop a new class of models satisfying mathematical requirements in particular.

2.1. Incompressibility

During the past decades three models of strain-energy functions for hyperelastic solid materials have usually been preferred in finite element applications, namely the models of Rivlin and Saunders (1951), Arruda and Boyce (1993) as well as Ogden (1972a). In the case of incompressibility these models comply with the representation shown in Table 1.

The *model of Rivlin and Saunders* represents a class of various models depending on the maximum order m and n as well as the material parameters $c_{ij} = 0$, which are prescribed in advance. For instance, $m = n = 1$ and $c_{00} = c_{11} = 0$ defines the classical Mooney-Rivlin model. In Hartmann (2001a,b) a variety of these models are summarized and studied with respect to identification of the material parameters c_{ij}, in particular their sensitivity with regard to the identification as well as requirements for the material parameters to ensure physically plausible curves. It is shown that for non-negative material parameters the well-known non-monotonous stress–strain curves in simple tension, biaxial tension, simple shear and tension–torsion tests cannot occur. For deformation processes mentioned before the identification of the material parameters leads to a linear least-square problem with non-negative solutions ($c_{ij} \geqslant 0$) which is simple to treat.

The *Arruda and Boyce model* is motivated by considerations on the microlevel and leads to two material parameters c and N which have to be positive in view of physical considerations. The factors d_i are fixed numbers defined a priori (they result from a Taylor expansion of the Langevin function). Obviously, the model is independent of the second invariant. The identification of material parameters leads to a non-linear least-square problem with a positive solution ($c > 0, N > 0$). Parameter identification is discussed, for example, by Przybylo and Arruda (1998) or Seibert and Schöche (2000).

Table 1
Strain-energy functions for incompressible hyperelastic materials

Rivlin/Saunders	$w_{\mathrm{RS}}(\mathrm{I_C}, \mathrm{II_C}) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} (\mathrm{I_C} - 3)^i (\mathrm{II_C} - 3)^j$ $\hat{w}_{\mathrm{RS}}(\mathbf{F}, \mathbf{F}^{-1}) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} (\|\mathbf{F}\|^2 - 3)^i (\|\mathbf{F}^{-1}\|^2 - 3)^j$
Arruda/Boyce	$w_{\mathrm{AB}}(\mathrm{I_C}) = c \sum_{i=1}^{m} d_i N^{1-i} (\mathrm{I}_{\mathrm{C}}^i - 3^i)$ $\hat{w}_{\mathrm{AB}}(\mathbf{F}) = c \sum_{i=1}^{m} d_i N^{1-i} (\|\mathbf{F}\|^{2i} - 3^i)$
Ogden	$w_{\mathrm{O}}(\mu_1, \mu_2, \mu_3) = \sum_{i=1}^{m} \frac{\gamma_i}{\alpha_i} (\mu_1^{\alpha_i/2} + \mu_2^{\alpha_i/2} + \mu_3^{\alpha_i/2} - 3)$ $\hat{w}_{\mathrm{O}}(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{m} \frac{\gamma_i}{\alpha_i} (\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3)$

The *Ogden model* depends on the eigenvalues μ_i of the right Cauchy–Green tensor, $\mathbf{C} = \sum_{i=1}^{3} \mu_i \vec{u}_i \otimes \vec{u}_i$, or on the eigenvalues $\lambda_i = \sqrt{\mu_i}$ of the right or left stretch tensors $\mathbf{U} = \sum_{i=1}^{3} \lambda_i \vec{u}_i \otimes \vec{u}_i$ or $\mathbf{V} = \sum_{i=1}^{3} \lambda_i \vec{v}_i \otimes \vec{v}_i$ (see Table 1), which result from the polar decomposition of the deformation gradient, $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$. $\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1}$ denotes the rotation tensor and $\vec{u}_i$ and $\vec{v}_i$ the eigenvectors of the right and left stretch tensors $\mathbf{U}$ and $\mathbf{V}$. In order to satisfy incompressibility, $\mu_3 = (\mu_1\mu_2)^{-1}$ must hold. γ_i and α_i represent material parameters. They should satisfy the inequalities $\gamma_i\alpha_i > 0$ (no sum over i) so that the constitutive model satisfies Hill's stability criteria to give physically plausible curves (Ogden, 1972a). According to Ciarlet (1988) the material parameters have to satisfy the conditions $\gamma_i > 0$ and $\alpha_i \geqslant 1$ in order to ensure polyconvexity and the coerciveness inequality, which is a stronger assumption in view of the range of the material parameters. Obviously, it fulfils Hill's stability criteria as well (Hill, 1970). The constitutive model is motivated for $n = 2$ by physical considerations on the microlevel by Kaliske and Heinrich (1999).

The identification of the material parameters of the Ogden model has been carried out by Twizell and Ogden (1983), Benjeddou et al. (1993), Gendy and Saleeb (2000), Przybylo and Arruda (1998) as well as Smeulders and Govindjee (1998). The identification leads to a non-linear least-square problem with non-linear inequality constraints. However, only a few of the aforementioned articles incorporate the inequality constraints of the material parameters. The number of Ogden terms is chosen with $m \leqslant 4$ at a maximum.

2.2. Weak compressibility

Usually the extension to nearly incompressible material behaviour is modelled using the multiplicative decomposition of the deformation gradient (2.1) into volume-preserving and volume-changing parts. Furthermore, it is assumed that the strain-energy function consists of two parts

$$\psi(J, \mathrm{I}_{\overline{\mathrm{C}}}, \mathrm{II}_{\overline{\mathrm{C}}}) = U(J) + w(\mathrm{I}_{\overline{\mathrm{C}}}, \mathrm{II}_{\overline{\mathrm{C}}}) \tag{2.8}$$

or

$$\hat{\psi}(J, \mathbf{F}, \operatorname{adj}\mathbf{F}) = U(J) + \hat{w}(J, \mathbf{F}, \operatorname{adj}\mathbf{F}) \tag{2.9}$$

so that the resulting stress state decomposes into a pure hydrostatic and a pure deviatoric part, see Eq. (3.14). The strain-energy functions in Table 1 are usually modified by exchanging the tensorial quantities, here the right Cauchy–Green tensor $\mathbf{C}$, with the unimodular tensor (2.4),

$$w_{\mathrm{RS}}(\mathrm{I}_{\overline{\mathrm{C}}}, \mathrm{II}_{\overline{\mathrm{C}}}) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} (\mathrm{I}_{\overline{\mathrm{C}}} - 3)^i (\mathrm{II}_{\overline{\mathrm{C}}} - 3)^j, \tag{2.10}$$

$$w_{\mathrm{AB}}(\mathrm{I}_{\overline{\mathrm{C}}}) = c \sum_{i=1}^{m} d_i N^{1-i} (\mathrm{I}_{\overline{\mathrm{C}}}^i - 3^i), \tag{2.11}$$

$$w_{\mathrm{O}}(\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3) = \sum_{i=1}^{m} \frac{\gamma_i}{\alpha_i} (\overline{\mu}_1^{\alpha_i/2} + \overline{\mu}_2^{\alpha_i/2} + \overline{\mu}_3^{\alpha_i/2} - 3), \quad \overline{\mu}_3 = (\overline{\mu}_1\overline{\mu}_2)^{-1} \tag{2.12}$$

or expressed in the variables $(J, \mathbf{F}, \operatorname{adj}\mathbf{F})$

$$\hat{w}_{\mathrm{RS}}(J, \mathbf{F}, \operatorname{adj}\mathbf{F}) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} (J^{-2/3} \|\mathbf{F}\|^2 - 3)^i (J^{-4/3} \|\operatorname{adj}\mathbf{F}\|^2 - 3)^j, \tag{2.13}$$

$$\hat{w}_{\mathrm{AB}}(J, \mathbf{F}) = c \sum_{i=1}^{m} d_i N^{1-i} \left(J^{-2i/3} \|\mathbf{F}\|^{2i} - 3^i \right), \tag{2.14}$$

$$\hat{w}_O(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3) = \sum_{i=1}^{m} \frac{\gamma_i}{\alpha_i}(\overline{\lambda}_1^{\alpha_i} + \overline{\lambda}_2^{\alpha_i} + \overline{\lambda}_3^{\alpha_i} - 3), \quad \overline{\lambda}_3 = (\overline{\lambda}_1\overline{\lambda}_2)^{-1}, \tag{2.15}$$

respectively.

In the literature the strain-energy function $U(J)$ is assumed to satisfy the physically plausible conditions $U(1) = 0$, $U(J)|_{J\to 0} = +\infty$ and $U(J)|_{J\to +\infty} = +\infty$. In Ehlers and Eipper (1998) ansatz (2.15) has been investigated. For simple tension tests they have shown that in the case of a determinant of the deformations gradient J, which is not close to one, non-physical behaviour of the models may occur. However, the reason for this and the desired a priori exclusion of these phenomena constitute an open problem. We look at these phenomena in Section 3.

2.3. Polyconvexity condition

In this part of the paper we investigate the polyconvexity conditions alluded to above and investigate the strain energies (2.13) and (2.14). Furthermore, new strain energy functions are proposed allowing for a proof of the existence of a solution of boundary-value problems. The modification in view of the isochoric and volumetric split of Ogden's strain-energy function (2.15) has already been investigated in terms of the existence of a solution (Charrier et al., 1988). Therefore, we restrict ourselves to the investigation of the special formulation using these isochoric invariants.

Whereas the whole formalism derived up to now could be based on considerations pertaining to the right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, the investigation of the *polyconvexity* condition is directly based on expressions defined on the deformation gradient $\mathbf{F}$. Therefore, in this subsection we use the deformation gradient $\mathbf{F}$ and the quantities $\operatorname{adj}\mathbf{F}$ and $\det \mathbf{F}$, i.e. the formulations (2.13) and (2.14).

Because the polyconvexity condition is connected to *convexity*, *quasi-convexity* and *ellipticity*, we have summarized the essential mathematical definitions in Appendices B and C, so that a self-explanatory text is obtained.

Since the following considerations are independent of the metric, we choose for simplicity a direct matrix notation. In this case we omit bold-face notation for the sake of easier readability. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3} = a^T b = b^T a$ symbolizing the scalar product on $\mathbb{R}^3$ with the norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3} = a^T a$. Furthermore, $\mathbb{M}^{3\times 3}$ denotes the set of real 3×3 matrices. Here, the standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle A, B\rangle = \operatorname{tr}(AB^T) = \operatorname{tr}(A^T B)$ with tr the trace operator and we have the norm $\|A\|^2_{\mathbb{R}^3} = \langle A, A\rangle$. For brevity we omit in the following the indices $\mathbb{R}^3$ and $\mathbb{M}^{3\times 3}$. $\operatorname{adj}A$ denotes the adjugate matrix, i.e. the matrix of transposed cofactors $\operatorname{cof}A$ such that $\operatorname{adj}A = (\det A)A^{-1} = (\operatorname{cof}A)^T$ if $A \in \mathrm{GL}(3, \mathbb{R})$, where $\mathrm{GL}(3, \mathbb{R})$ is the set of invertible 3×3 matrices. The identity matrix on $\mathbb{M}^{3\times 3}$ will be denoted by $\mathbb{1}$ so that $\operatorname{tr}A = \langle A, \mathbb{1}\rangle$ holds. A lower dot in $a = A.b$ symbolizes the application of $A \in \mathbb{M}^{3\times 3}$ onto $b \in \mathbb{R}^3$ yielding the vector $a \in \mathbb{R}^3$. Furthermore, we need the first and second Fréchet derivatives $Df(A).H$ and $D^2 f(A).(H, H)$. Subsequently, we need the sets $\mathbb{S}\mathrm{ym}$ and $\mathbb{S}\mathrm{ym}^+$ denoting the sets of symmetric and symmetric positive definite 3×3 matrices, respectively.

First of all, we emphasize the property of unimodularity for $\overline{C} = \overline{F}^T\overline{F}$, namely $\det \overline{C} = 1$ with $\overline{C}$ in Eq. (2.4). Since we intend to investigate strain energies of the form

$$W(F) = U(\det F) + W_{\mathrm{iso}}\left(\frac{F^T F}{(\det F^T F)^{1/3}}\right),$$

i.e. the free energy decomposes additively into two terms resulting from purely isochoric and volumetric deformations. We will show that this ansatz is compatible, under certain circumstances, with the requirement of polyconvexity. To this end, we first study the isochoric part of the strain energy function and then we focus our attention to the (simpler) volumetric part.

2.3.1. Isochoric part of the strain energy function

Let us start with a preliminary clarification since the determinant of the deformation gradient has to be assumed to be positive. Let, for example, $W_1(A) = \langle A, \mathbb{1}\rangle$, and define

$$\text{iso}\,(F) \equiv \overline{C} = \frac{F^{\mathrm{T}}F}{(\det(F^{\mathrm{T}}F))^{1/3}}.$$

Then

$$W_{\text{iso}}\left(\frac{F^{\mathrm{T}}F}{(\det(F^{\mathrm{T}}F))^{1/3}}\right) = W_1(\text{iso}\,(F)) = \begin{cases} \frac{\|F\|^2}{(\det F)^{2/3}} & \det F > 0, \\ \infty & \det F \leqslant 0 \end{cases}$$

and $W(F) = W_1(\text{iso}\,(F))$ is a polyconvex function (however, not of the additive type, Corollary C.2). For the remainder let us agree to extend functions W which are naturally only defined on the set $\det F > 0$ to $\mathbb{M}^{3\times 3}$ by setting $W = \infty$ for arguments with $\det F \leqslant 0$. With such an extension it is clear that W can never be convex, for it is supported on a non-convex set only. However, this extension is compatible with the requirement of polyconvexity since

$$P(x) = \begin{cases} f(x) & x > 0, \\ \infty & x \leqslant 0 \end{cases} \tag{2.16}$$

is a convex function whenever f is convex on $\mathbb{R}^+$.

In the following, we state several lemmas. The first one is connected to the basic invariant $\mathrm{I}_{\overline{\mathrm{C}}}$. The second one studies generalized polyconvex strain energy terms, which are followed by two essential terms satisfying a stress-free reference configuration. Lastly, we show that in the case of the generalized polynomial-type elasticity (2.13) for various terms ellipticity may be lost and an existence proof based on our methods cannot be given.

We start with the investigation of strain energy functions depending on the first invariant $\mathbf{I}_{\overline{\mathbf{C}}}$ of unimodular right Cauchy tensor $\overline{\mathbf{C}}$, here expressed by the deformation gradient $\mathbf{F}$:

Lemma 2.1 (Isochoric terms). *Let the strain energy be of the type* $W(F) = \|F\|^2/(\det F)^{2/3}$. *Then* W *is polyconvex.*

This can be proved as follows:

Proof. First, we investigate the convexity of the function $P : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $P(x, y) = f(x)g(y)$. The matrix of second derivatives is of course

$$D^2P(x,y) = \begin{bmatrix} f''(x)g(y) & f'(x)g'(y) \\ f'(x)g'(y) & f(x)g''(y) \end{bmatrix}.$$

If f, g are positive, smooth and convex, then we have $f''(x)g(y) \geqslant 0$ and $\det D^2P(x,y) = f''(x)g(y)f(x)g''(y) - (f'(x)g'(x))^2$. Note that P is convex, if D^2P is positive definite by Lemma B.2. In our situation D^2P is positive definite if $f''(x) \cdot g(y) \geqslant 0$ and $\det D^2P(x,y) \geqslant 0$. Thus we must guarantee that $f''(x)g(y)f(x)g''(y) \geqslant (f'(x)g'(x))^2$.

Let $\alpha > 0$ and $p \geqslant 2$. We choose $f(x) = x^{-\alpha}$ and $g(y) = y^p$. Then

$$f''(x)g(y)f(x)g''(y) = \alpha(\alpha+1)x^{-(2+\alpha)}y^p x^{-\alpha}p(p-1)y^{p-2}$$

and

$$(f'(x)g'(x))^2 = (-\alpha x^{-(\alpha+1)}py^{p-1})^2 = \alpha^2 x^{-2(\alpha+1)}p^2 y^{2(p-1)}.$$

We arrive at the condition that

$$\frac{\alpha+1}{\alpha} \geqslant \frac{p}{p-1}.$$

The larger one chooses p, the better for the choice of α. Notably, $P(x,y) = (1/x^{\alpha}) \cdot y^{p}$ is convex for $\alpha = 2/3$ and $p = 2$. We set

$$\widehat{W}(F,\widetilde{J}) = P(\widetilde{J}, \|F\|) = \frac{\|F\|^2}{\widetilde{J}^{2/3}}.$$

We check the convexity of $\widehat{W}(F,\widetilde{J})$. Thus

$$\widehat{W}(\lambda F_1 + (1-\lambda)F_2, \lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2) = P(\lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2, \|\lambda F_1 + (1-\lambda)F_2\|) = \frac{\|\lambda F_1 + (1-\lambda)F_2\|^2}{(\lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2)^{2/3}}$$

and the monotonicity of the square for positive arguments yields

$$\begin{aligned}\widehat{W}(\lambda F_1 + (1-\lambda)F_2, \lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2) &\leqslant \frac{(\lambda\|F_1\| + (1-\lambda)\|F_2\|)^2}{(\lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2)^{2/3}} \\ &= P(\lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2, \lambda\|F_1\| + (1-\lambda)\|F_2\|).\end{aligned}$$

Since by assumption P is convex, we obtain

$$\begin{aligned}\widehat{W}(\lambda F_1 + (1-\lambda)F_2, \lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2) &\leqslant \lambda P(\widetilde{J}_1, \|F_1\|) + (1-\lambda)P(\widetilde{J}_2, \|F_2\|) \\ &= \lambda\widehat{W}(F_1, \widetilde{J}_1) + (1-\lambda)\widehat{W}(F_2, \widetilde{J}_2).\end{aligned}$$

Now recall the extension of W to all of $\mathbb{M}^{3\times3}$ and use (2.16). Thus we have shown that $\widehat{W}$ is convex on the convex set $\mathbb{M}^{3\times3} \times \mathbb{R}^{+}$ and convexly extended to $\mathbb{M}^{3\times3} \times \mathbb{R}$. The proof is complete. For a proof also compare with Charrier et al. (1988) or Dacorogna (1989, p. 140). □

Since we are interested in the investigation of the modified generalized polynomial-type elasticity (2.10) or (2.13), respectively, we look at the following specific terms:

Lemma 2.2 (Special polyconvex terms). *Let $F \in \mathbb{M}^{3\times3}$. Then each of the following terms is polyconvex*:

$$(1)\quad F \mapsto \left(\frac{\|F\|^2}{(\det F)^{2/3}} - 3\right)^{i} = (\operatorname{tr}\overline{C} - 3)^{i}, \quad i \geqslant 1.$$

$$(2)\quad F \mapsto \left(\frac{\|\operatorname{adj} F\|^3}{(\det F)^2} - 3\sqrt{3}\right)^{j} = ((\operatorname{tr}\operatorname{adj}\overline{C})^{3/2} - 3\sqrt{3})^{j}, \quad j \geqslant 1.$$

Proof

(1) We have already checked in Lemma 2.1 that the expression $\|F\|^2/(\det F)^{2/3}$ is polyconvex, hence there exists a convex function $P(F, \det F) = \|F\|^2/(\det F)^{2/3}$. Note that in view of the estimates for the invariants Corollary A.3 of Appendix A we know that $P(F, \det F) - 3 \geqslant 0$. We define the function $[a]_{+} = \max\{a, 0\}$. Note that $x \mapsto \max\{f(x), 0\}$ is convex if f is convex. Then

$$\left(\frac{\|F\|^2}{(\det F)^{2/3}} - 3\right)^i = [P\underbrace{(F, \det F)}_{X\in\mathbb{R}^{10}} - 3]_+^i.$$

P is convex in F and $x \to x^i$, $i \geqslant 1$, is monotonically increasing for positive values and convex, hence $[P(X) - 3]_+^i$ is altogether convex in X which is however the polyconvexity of $F \mapsto [P(F, \det F) - 3]_+^i$. Since this last expression coincides with

$$\left(\frac{\|F\|^2}{(\det F)^{2/3}} - 3\right)^i$$

the polyconvexity is proved.

(2) We know already that $(\|\mathrm{adj}\, F\|^3/(\det F)^2) - 3\sqrt{3}$ is polyconvex since the exponents verify the decisive inequality $(\alpha + 1)/\alpha \geqslant p/(p-1)$. Moreover, $(\|\mathrm{adj}\, F\|^3/(\det F)^2) - 3\sqrt{3} \geqslant 0$ with Lemma A.3. Exactly the same reasoning applies now as before. □

A generalization of the above mentioned strain energy functions yield the more general class of isochoric strain energy terms by the following corollary.

Corollary 2.3. *Let $F \in \mathbb{M}^{3\times 3}$. Then each of the following more general terms is polyconvex*:

$$(1) \quad F \mapsto \left(\frac{\|F\|^{2k}}{(\det F)^{2k/3}} - 3^k\right)^i, \quad i \geqslant 1, k \geqslant 1.$$

$$(2) \quad F \mapsto \left(\frac{\|\mathrm{adj}\, F\|^{3k}}{(\det F)^{2k}} - (3\sqrt{3})^k\right)^j, \quad j \geqslant 1, k \geqslant 1.$$

$$(3) \quad F \mapsto \exp\left[\left(\frac{\|F\|^{2k}}{(\det F)^{2k/3}} - 3^k\right)^i\right] - 1, \quad i \geqslant 1, k \geqslant 1.$$

$$(4) \quad F \mapsto \exp\left[\left(\frac{\|\mathrm{adj}\, F\|^{3k}}{(\det F)^{2k}} - (3\sqrt{3})^k\right)^j\right] - 1, \quad j \geqslant 1, k \geqslant 1.$$

In order to prove the aforementioned corollary one has to apply the same ideas as before and note that exp is a convex monotonically increasing function, so we may apply Lemma B.7.

Since we are interested in the generalized polynomial-type elasticity, we look at the terms of the second invariant $\mathrm{II}_{\overline{\mathbf{C}}} = J^{-4/3}\|\mathrm{adj}\,\mathbf{F}\|^2$ in Eq. (2.10) or (2.13). These terms are not polyconvex, i.e. the existence of a solution of a boundary-value problem cannot be guaranteed:

Lemma 2.4 (Non-ellipticity of mixed terms). *The following terms are non-elliptic hence not polyconvex:*

$$W(F) = \left(\frac{\|F\|^2}{(\det F)^{2/3}} - 3\right)^i \left(\frac{\|\mathrm{adj}\, F\|^3}{(\det F)^2} - 3\sqrt{3}\right)^j, \quad i, j \geqslant 1.$$

Moreover, the term

$$F \mapsto \left(\frac{\|\mathrm{adj}\, F\|^2}{(\det F)^{4/3}} - 3 \right)^i, \quad i \geqslant 1$$

is non-elliptic, hence it cannot be polyconvex, even so $(\|\mathrm{adj}\, F\|^2/(\det F)^{4/3}) - 3 \geqslant 0$ *in the light of Corollary A.3. Here the term* $\|\mathrm{adj}\, F\|^2/(\det F)^{4/3}$ *itself does not have the right exponents for polyconvexity* (Charrier et al., 1988).

Proof. We let $i, j = 1$ and consider the eigenvalue representation of $W(F)$:

$$W(F) = \left(\frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{(\lambda_1\lambda_2\lambda_3)^{2/3}} - 3 \right)^i \left(\frac{[(\lambda_1\lambda_2)^2 + (\lambda_2\lambda_3)^2 + (\lambda_1\lambda_3)^2]^{3/2}}{(\lambda_1\lambda_2\lambda_3)^2} - 3\sqrt{3} \right)^j = \Phi(\lambda_1, \lambda_2, \lambda_3).$$

We take a deformation with deformation gradient $F = \mathrm{diag}\,(0.1, 10, t)$ with $t \in \mathbb{R}^+$. If $W(F)$ is rank-one convex, then

$$\Phi(0.1, 10, t) = \left(\frac{100.01 + t^2}{t^{2/3}} - 3 \right)^i \left(\frac{[1 + 100.01t^2]^{3/2}}{t^2} - 3\sqrt{3} \right)^j$$

should be convex, according to Theorem C.5. However, this is not the case, as can easily be verified. Typically, convexity in t (hence ellipticity with respect to F) is lost for extreme deformations only. □

Table 2 summarizes new classes of polyconvex strain energy functions in terms of tensorial quantities and in respect to the decomposition into isochoric and volumetric parts.

Lemma 2.4 has shown that the modified generalized polynomial-type elasticity cannot be polyconvex as a result of the choice of the terms $(\mathrm{II}_{\overline{\mathbf{C}}} - 3)^j$. If we modify these strain-energy functions into $\mathrm{II}_{\overline{\mathbf{C}}}^{3/2} - 3\sqrt{3}$—see Corollary 2.3, proposal 2, in the case of $k = 1$—polyconvexity is only sati2sfied if products $(\mathrm{II}_{\overline{\mathbf{C}}} - 3)^i(\mathrm{II}_{\overline{\mathbf{C}}}^{3/2} - 3\sqrt{3})^j$ do not occur, i.e. the first and second invariant are decoupled, see Lemma 2.4 as well. Furthermore, we point out in view of Corollary 2.3, proposal 1, that in the case of $i = 1$ the isochoric part of the strain energy of Arruda and Boyce is polyconvex.

2.3.2. Volumetric part of the strain-energy function

The volumetric part of the strain energy function $U(J)$ in ansatz (2.8) or (2.9) has merely to be convex in the variable $J = \det \mathbf{F}$. In Table 3 various new convex strain energy functions are proposed. Of course, they have to be modified in order to satisfy the requirement of a stress-free reference configuration.

In the framework of the finite element method one mostly uses one term of the volumetric strain energy, $U(J) = K\widehat{U}(J)$, where K represents the compression modulus and $\widehat{U}(J)$ the principle function of the determinant J. From the physical point of view we should fulfill a energy- and stress-free reference configuration $U(1) = 0$ and $U'(1) = 0$. In the case of $U''(1) = K$, i.e. $\widehat{U}''(1) = 1$, K can be interpreted as the compression modulus of linear elasticity. The convexity requirement implies $U(J) \to \infty$ for $J \to 0$ and

Table 2
Polyconvex isochoric strain energy terms

$\varphi_1(\mathrm{I}_{\overline{\mathrm{C}}}) \equiv \left(\mathrm{I}_{\overline{\mathrm{C}}}^k - 3^k\right)^i$	$i \geqslant 1,\ k \geqslant 1$
$\varphi_2(\mathrm{II}_{\overline{\mathrm{C}}}) \equiv \left(\mathrm{II}_{\overline{\mathrm{C}}}^{3k/2} - (3\sqrt{3})^k\right)^j$	$j \geqslant 1,\ k \geqslant 1$
$\varphi_3(\varphi_1(\mathrm{I}_{\overline{\mathrm{C}}})) \equiv \exp(\varphi_1(\mathrm{I}_{\overline{\mathrm{C}}})) - 1$	$i \geqslant 1,\ k \geqslant 1$
$\varphi_4(\varphi_2(\mathrm{I}_{\overline{\mathrm{C}}})) \equiv \exp(\varphi_2(\mathrm{II}_{\overline{\mathrm{C}}})) - 1$	$j \geqslant 1,\ k \geqslant 1$

Table 3
New convex isochoric strain energy functions

$(J^{2p} + J^{-2p} - 2)^k$	$p \geqslant 1/2,\ k \geqslant 1$
$(J-1)^k$	$k > 1$
$J^2 - 2\ln J + 4(\ln J)^2$	

Table 4
Volumetric strain energy functions of the literature

	$\widehat{U}(J)$	$\widehat{U}'(J)$	$\widehat{U}''(J)$	Reference
1.	$\frac{1}{2}(J-1)^2$	$J-1$	1	
2.	$\frac{1}{4}\left((J-1)^2 + (\ln J)^2\right)$	$\frac{1}{2}\left(J - 1 + \frac{1}{J}\ln J\right)$	$\frac{1}{2J^2}(1 + J^2 - \ln J)$	Simo and Taylor (1982)
3.	$\frac{1}{2}(\ln J)^2$	$\frac{1}{J}\ln J$	$\frac{1}{J^2}(1 - \ln J)$	Simo et al. (1985)
4.	$\frac{1}{\beta^2}\left(\frac{1}{J^\beta} - 1 + \beta\ln J\right)$	$\frac{1}{\beta}\left(\frac{1}{J} - \frac{1}{J^{1+\beta}}\right)$	$\frac{1}{\beta}\left(\frac{1}{J^{2+\beta}}(1 + \beta - J^\beta)\right)$	Ogden (1972b)
5.	$\frac{1}{4}(J^2 - 1 - 2\ln J)$	$\frac{1}{2}\left(J - \frac{1}{J}\right)$	$\frac{1}{2}\left(1 + \frac{1}{J^2}\right)$	Simo and Taylor (1991)
6.	$J - \ln J - 1$	$1 - \frac{1}{J}$	$\frac{1}{J^2}$	Miehe (1994)
7.	$J^\beta(\beta\ln J - 1) + 1$	$\beta^2\frac{1}{J^{1-\beta}}\ln J$	$\beta^2 J^{\beta-2}(1 + (\beta - 1)\ln J)$	Hartmann (2002)
8.	$J\ln J - J + 1$	$\ln J$	$\frac{1}{J}$	Liu et al. (1994)
9.	$\frac{1}{32}(J^2 - J^{-2})^2$	$\frac{1}{8}\left(J^3 - \frac{1}{J^5}\right)$	$\frac{1}{8}\left(5\frac{1}{J^6} + 3J^2\right)$	ANSYS (2000)
10.	$\frac{J}{\beta}\left(1 - \frac{J^{-\beta}}{1-\beta}\right) + \frac{1}{\beta-1}$	$\frac{1}{\beta}(1 - J^{-\beta})$	$J^{-(1+\beta)}$	Murnaghan (1951, S. 68)
11.	$\frac{1}{50}(J^5 + J^{-5} - 2)$	$\frac{1}{10}(J^4 - J^{-6})$	$\frac{1}{10}(4J^4 + 6J^{-7})$	

$J \to \infty$ as well as $U''(J) \geqslant 0$, so that a volumetric compression or stretch yields hydrostatic pressure or tension. In Table 4 models of the literature are summarized. Model 11 is a particular version of the first proposal of Table 2 for $k = 1$ and $p = 5/2$, which is used in the forthcoming section.

Figs. 1 and 2 show the behaviour of the models 1–3, 5, 6, 8, 9 as well as 11 in the region $0 < J \leqslant 5$. Model 1 represents a linear approximation of hydrostatic stresses and is included in the generalized form, no. 1, in Table 3, $(\mathrm{tr}\,\mathbf{T})/3 = U'(J)$, see Eq. (3.14). However, model 1 has a finite limit in the case of $\lim_{J\to 0}\widehat{U}(J) = 1/2$ or $\lim_{J\to 0}\widehat{U}'(J) = -1$, which is not plausible for higher volumetric deformations. The

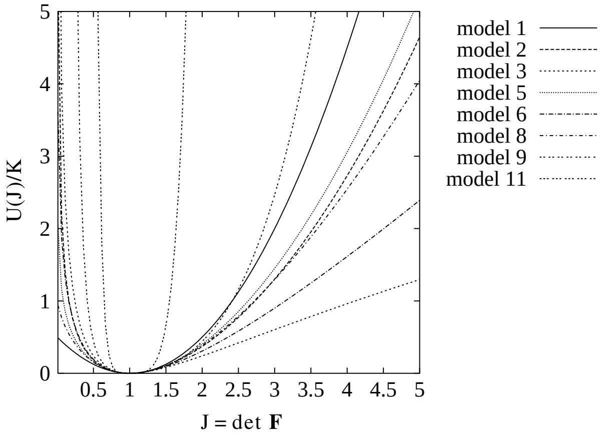

Fig. 1. Behaviour of the strain energy functions in Table 4.

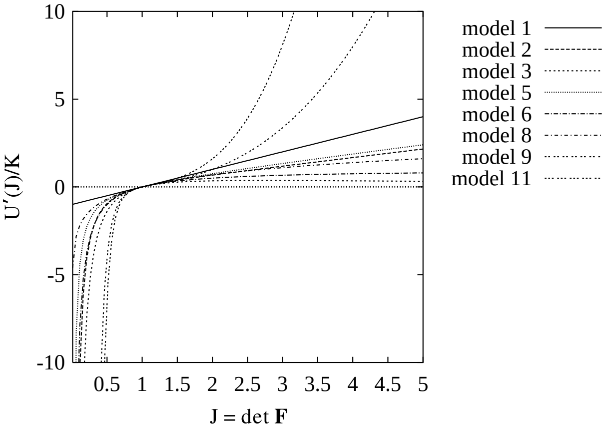

Fig. 2. Behaviour of the hydrostatic stresses in view of the volumetric strain energy functions of Table 4.

extension by the natural logarithm in model 2 corrects this characteristic. If one uses only the correction term, see model 3, convexity is violated for $J > e = 2.718\ldots$, i.e. $\widehat{U}''(J) < 0$. Fig. 2 shows a decreasing stress curve during volumetric tension. Models 5 and 6 are included in Ogden's model, here model 4 for $\beta = -2$ and $\beta = -1$. The case $\beta = 1$ is applied in Ehlers and Eipper (1998). Model 8 is, however, incorporated for $\beta = 1$ in model 7. Obviously, convexity is not satisfied for all β in model 7. In this work we have modified models 8 and 9 in view of the original literature in order to satisfy condition $\widehat{U}''(1) = 1$. Model 10 of Murnaghan (1951, S. 68) is originally developed in terms of hydrostatic pressure. Here, we developed the strain-energy concerned by integration. Our proposal of model 11 has some advantageous properties in view of physical plausible tensile tests, see later.

3. Investigation of the proposed strain-energy function

The aforementioned mathematical studies lead, for example, to the polyconvex strain energy

$$\psi(J, \mathrm{I}_{\overline{\mathbf{C}}}, \mathrm{II}_{\overline{\mathbf{C}}}) = U(J) + w(\mathrm{I}_{\overline{\mathbf{C}}}, \mathrm{II}_{\overline{\mathbf{C}}}) \tag{3.1}$$

with

$$U(J) = \frac{K}{50}(J^5 + J^{-5} - 2), \tag{3.2}$$

$$w(\mathrm{I}_{\overline{\mathbf{C}}}, \mathrm{II}_{\overline{\mathbf{C}}}) = \alpha(\mathrm{I}_{\overline{\mathbf{C}}}^3 - 3^3) + \sum_{i=1}^{m} c_{i0}(\mathrm{I}_{\overline{\mathbf{C}}} - 3)^i + \sum_{j=1}^{n} c_{0j}(\mathrm{II}_{\overline{\mathbf{C}}}^{3/2} - 3\sqrt{3})^j \tag{3.3}$$

which satisfies coercivity as well (see Appendix D).

Remark 1. A further mathematical notion is called "coercivity". Coercivity is a condition imposed on the growth of the strain energy for deformation gradients in the range of finite deformations. It is a necessary part of the existence proof via the direct methods of variations. For additional comments see Ciarlet (1988). A concrete strain energy formulation, which satisfies this condition as well, is given in Eqs. (3.2) and (3.3). The proof of its coercivity is given in Appendix D.

First of all, we have to mention the problem of material parameter identification. Since not all deformations for compressible solids yield analytical solutions, the identification problem for the strain energy (3.1) is non-linear—see, for example, equation system (3.22). If the compressible part of the strain energy is only chosen with a view to improving the numerical calculations by means of the finite element method and the material under investigation is very close to the incompressible case, then the parameter estimation can be carried out by identifying the parameters $\alpha > 0$, $c_{i0} \geqslant 0$, $i = 1, \ldots, m$, and $c_{0j} \geqslant 0$, $j = 1, \ldots, m$, using the assumption of incompressibility. This yields in the cases of simple tension, pure and simple shear, biaxial tension and combined tension–torsion tests analytical solutions which are linear in the material parameters. In this case one has to apply a linear least-square method with non-negative solutions (see e.g., Hartmann, 2001a,b). Here, we have to emphasize that during the identification process the material parameter α has to be different to zero in order to satisfy the coerciveness inequality.

Here, we choose one of the simplest constitutive models with $U(J)$ of Eq. (3.2) and with $m = n = 1$ of Eq. (3.3),

$$w(\mathrm{I}_{\overline{\mathbf{C}}}, \mathrm{II}_{\overline{\mathbf{C}}}) = \alpha(\mathrm{I}^3_{\overline{\mathbf{C}}} - 27) + c_{10}(\mathrm{I}_{\overline{\mathbf{C}}} - 3) + c_{01}(\mathrm{II}^{3/2}_{\overline{\mathbf{C}}} - 3\sqrt{3}), \tag{3.4}$$

i.e. we are interested in the identification of the material parameters α, c_{10} and c_{01}. For the identification process we use the experimental data of Haupt and Sedlan (2001) of a tension, a pure torsion and two combined tension–torsion tests applying a particular weighting technique, see Hartmann (2001b). This yields the material parameters $\alpha = 0.00367$ MPa, $c_{01} = 0.1958$ MPa and $c_{10} = 0.1788$ MPa. Then we obtain the S-shaped curve in a uniaxial tension-compression diagram (see Fig. 3).

For these material parameters, we later discuss the behaviour of the model in the nearly incompressible case. To this end, we consider in a first step the general stress state which is calculated as follows: the second Piola–Kirchhoff tensor $\widetilde{\mathbf{T}}$ is defined by

$$\widetilde{\mathbf{T}} = 2\frac{\mathrm{d}\psi}{\mathrm{d}\mathbf{C}} = 2\left(\frac{\mathrm{d}U((\det \mathbf{C})^{1/2})}{\mathrm{d}\mathbf{C}} + \frac{\mathrm{d}w(\overline{\mathbf{C}}(\mathbf{C}))}{\mathrm{d}\mathbf{C}}\right). \tag{3.5}$$

The derivative $\mathrm{d}U/\mathrm{d}\mathbf{C}$ is given by

$$\frac{\mathrm{d}U((\det \mathbf{C})^{1/2})}{\mathrm{d}\mathbf{C}} = \frac{1}{2}JU'(J)\mathbf{C}^{-1}, \tag{3.6}$$

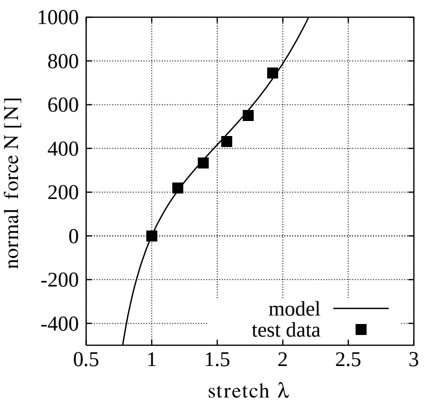

Fig. 3. Uniaxial stress-stretch behaviour (incompressibility).

where we make use of $J = (\det \mathbf{C})^{1/2}$. The derivative $\mathrm{d}w/\mathrm{d}\mathbf{C}$ is calculated by means of

$$\frac{\mathrm{d}w(\overline{\mathbf{C}}(\mathbf{C}))}{\mathrm{d}\mathbf{C}} = \left[\frac{\mathrm{d}\overline{\mathbf{C}}}{\mathrm{d}\mathbf{C}}\right]^{\mathrm{T}} \frac{\mathrm{d}w}{\mathrm{d}\overline{\mathbf{C}}} \tag{3.7}$$

with

$$\left[\frac{\mathrm{d}\overline{\mathbf{C}}}{\mathrm{d}\mathbf{C}}\right]^{\mathrm{T}} = (\det \mathbf{C})^{-1/3}\left[\mathscr{I} - \frac{1}{3}(\mathbf{C}^{-1} \otimes \mathbf{C})\right] = J^{-2/3}\left[\mathscr{I} - \frac{1}{3}(\overline{\mathbf{C}}^{-1} \otimes \overline{\mathbf{C}})\right]. \tag{3.8}$$

$\mathscr{I}$ denotes the fourth-order identity tensor, $\mathscr{I}\mathbf{A} = \mathbf{A}$. Obviously, $\mathbf{C}^{-1} \otimes \mathbf{C} = \overline{\mathbf{C}}^{-1} \otimes \overline{\mathbf{C}}$ holds. Since we have the particular dependence of the invariants, the derivative $\mathrm{d}w/\mathrm{d}\mathbf{C}$ results from the chain rule

$$\frac{\mathrm{d}w}{\mathrm{d}\overline{\mathbf{C}}} = \frac{\partial w}{\partial \mathrm{I}_{\overline{\mathbf{C}}}} \frac{\mathrm{d}\mathrm{I}_{\overline{\mathbf{C}}}}{\mathrm{d}\overline{\mathbf{C}}} + \frac{\partial w}{\partial \mathrm{II}_{\overline{\mathbf{C}}}} \frac{\mathrm{d}\mathrm{II}_{\overline{\mathbf{C}}}}{\mathrm{d}\overline{\mathbf{C}}} = (w_1 + w_2 \mathrm{I}_{\overline{\mathbf{C}}})\mathbf{I} - w_2 \overline{\mathbf{C}} \tag{3.9}$$

with

$$w_1(\mathrm{I}_{\overline{\mathbf{C}}}, \mathrm{II}_{\overline{\mathbf{C}}}) = \frac{\partial w}{\partial \mathrm{I}_{\overline{\mathbf{C}}}} = 3\alpha \mathrm{I}_{\overline{\mathbf{C}}}^2 + \sum_{i=1}^{m} c_{i0} i (\mathrm{I}_{\overline{\mathbf{C}}} - 3)^{i-1} \tag{3.10}$$

and

$$w_2(\mathrm{I}_{\overline{\mathbf{C}}}, \mathrm{II}_{\overline{\mathbf{C}}}) = \frac{\partial w}{\partial \mathrm{II}_{\overline{\mathbf{C}}}} = \sum_{j=1}^{n} c_{0j} j \frac{3}{2} \mathrm{II}_{\overline{\mathbf{C}}}^{1/2} (\mathrm{II}_{\overline{\mathbf{C}}}^{3/2} - 3\sqrt{3})^{j-1}. \tag{3.11}$$

Now, we arrive at two parts of the stress state:

$$\widetilde{\mathbf{T}} = \widetilde{\mathbf{T}}_{\mathrm{vol}} + \widetilde{\mathbf{T}}_{\mathrm{iso}} \tag{3.12}$$

$$= JU'(J)\mathbf{C}^{-1} + 2J^{-2/3}\left((w_1 + w_2 \mathrm{I}_{\overline{\mathbf{C}}})\mathbf{I} - w_2 \overline{\mathbf{C}} - \tfrac{1}{3}(w_1 \mathrm{I}_{\overline{\mathbf{C}}} + 2 w_2 \mathrm{II}_{\overline{\mathbf{C}}})\overline{\mathbf{C}}^{-1}\right). \tag{3.13}$$

Additionally, we remark that the push-forward of the second Piola–Kirchhoff tensor to the current configuration, represented by the Cauchy stress tensor $\mathbf{T} = J^{-1}\mathbf{F}\widetilde{\mathbf{T}}\mathbf{F}^{\mathrm{T}}$, leads to

$$\mathbf{T} = U'(J)\mathbf{I} + \frac{2}{J}\left(\frac{\mathrm{d}w}{\mathrm{d}\overline{\mathbf{B}}}\overline{\mathbf{B}}\right)^{D}, \tag{3.14}$$

where it becomes obvious that the decomposition of the strain-energy function yields in a natural way purely hydrostatic and pure deviatoric stress states caused by the specific form of the strain-energy function. The superscript D symbolizes the deviator operator, $\mathbf{A}^D = \mathbf{A} - (1/3)(\mathrm{tr}\,\mathbf{A})\mathbf{I}$.

The tangent operator concerned

$$\widetilde{\mathscr{C}} = 4\frac{\mathrm{d}^2\psi}{\mathrm{d}\mathbf{C}\,\mathrm{d}\mathbf{C}} = 2\frac{\mathrm{d}\widetilde{\mathbf{T}}}{\mathrm{d}\mathbf{C}} = \widetilde{\mathscr{C}}_{\mathrm{vol}} + \widetilde{\mathscr{C}}_{\mathrm{iso}} \tag{3.15}$$

which decomposes additively into a volumetric and an isochoric part, has the representation

$$\widetilde{\mathscr{C}}_{\mathrm{vol}} = J\left[(U'(J) + JU''(J))\mathbf{C}^{-1} \otimes \mathbf{C}^{-1} - 2U'(J)[\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}]^{\mathrm{T}_{23}}\right], \tag{3.16}$$

$$\widetilde{\mathscr{C}}_{\mathrm{iso}} = 4J^{-4/3}\left[\mathscr{I} - \frac{1}{3}\overline{\mathbf{C}}^{-1} \otimes \overline{\mathbf{C}}\right]\frac{\mathrm{d}^2 w}{\mathrm{d}\overline{\mathbf{C}}\,\mathrm{d}\overline{\mathbf{C}}}\left[\mathscr{I} - \frac{1}{3}\overline{\mathbf{C}} \otimes \overline{\mathbf{C}}^{-1}\right] - \frac{2J^{-2/3}}{3}\left[\widetilde{\mathbf{T}}_{\mathrm{iso}} \otimes \overline{\mathbf{C}}^{-1} + \overline{\mathbf{C}}^{-1} \otimes \widetilde{\mathbf{T}}_{\mathrm{iso}}\right]$$
$$+ \frac{4J^{-4/3}}{3}\left(\overline{\mathbf{C}} \cdot \frac{\mathrm{d}w}{\mathrm{d}\overline{\mathbf{C}}}\right)\left[\left[\overline{\mathbf{C}}^{-1} \otimes \overline{\mathbf{C}}^{-1}\right]^{\mathrm{T}_{23}} - \frac{1}{3}\overline{\mathbf{C}}^{-1} \otimes \overline{\mathbf{C}}^{-1}\right] \tag{3.17}$$

with

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\overline{\mathbf{C}}\,\mathrm{d}\overline{\mathbf{C}}} = \left(w_{11} + 2w_{12}\mathrm{I}_{\overline{\mathrm{C}}} + w_{22}\mathrm{I}_{\overline{\mathrm{C}}}^2 + w_2\right)\mathbf{I}\otimes\mathbf{I} - (w_{12} + \mathrm{I}_{\overline{\mathrm{C}}}w_{22})\left[\mathbf{I}\otimes\overline{\mathbf{C}} + \overline{\mathbf{C}}\otimes\mathbf{I}\right] - w_2\mathscr{I} + w_{22}\overline{\mathbf{C}}\otimes\overline{\mathbf{C}}, \tag{3.18}$$

see, for example, Holzapfel (2000). Here, we introduce the following abbreviations: the superscript T_{23} denotes the transposition of the second and third index leading to $[\mathbf{A}\otimes\mathbf{B}]^{\mathrm{T}_{23}}\mathbf{C} = \mathbf{A}\mathbf{C}\mathbf{B}^{\mathrm{T}}$, as well as $w_{i1} = \partial w_i/\partial \mathrm{I}_{\overline{\mathrm{C}}}$ and $w_{i2} = \partial w_i/\partial \mathrm{II}_{\overline{\mathrm{C}}}$.

Now, in the case of our particular isochoric strain-energy function (3.4) we are interested in the limit of small deformations and connecting the material parameters to the material parameters of linear elasticity, namely the compression modulus K and the shear modulus G. Near the reference state, $\widetilde{\mathscr{C}}|_{\mathbf{F}=\mathbf{I}}$, we obtain from Eqs. (3.16) and (3.17) the elasticity tensor

$$\widetilde{\mathscr{C}}|_{\mathbf{F}=\mathbf{I}} = K\mathbf{I}\otimes\mathbf{I} + 2G\left[\mathscr{I} - \tfrac{1}{3}\mathbf{I}\otimes\mathbf{I}\right] \tag{3.19}$$

with

$$G = 54\alpha + 2c_{10} + 3\sqrt{3}c_{01}. \tag{3.20}$$

The compression modulus K of the finite elasticity model coincides with K of the linear elastic case in a natural way.

Now, we investigate the simple tension problem with a deformation gradient $\mathbf{F} = \lambda\vec{e}_1\otimes\vec{e}_1 + \lambda_Q\vec{e}_2\otimes\vec{e}_2 + \lambda_Q\vec{e}_3\otimes\vec{e}_3$, where λ denotes the axial stretch and λ_Q the transversal stretch. The stresses in transversal direction are zero. Thus, Eq. (3.13) leads to the two equations

$$\widetilde{T}_{11} = f(\lambda,\lambda_Q), \tag{3.21}$$

$$0 = g(\lambda,\lambda_Q) \tag{3.22}$$

for given λ, with

$$f(\lambda,\lambda_Q) \equiv JU'(J)\lambda^{-2} + 2J^{-2/3}\left(w_1 + w_2\mathrm{I}_{\overline{\mathrm{C}}} - w_2\lambda^2 J^{-2/3} - \tfrac{1}{3}(w_1\mathrm{I}_{\overline{\mathrm{C}}} + 2w_2\mathrm{II}_{\overline{\mathrm{C}}})\lambda^{-2}J^{2/3}\right), \tag{3.23}$$

$$g(\lambda,\lambda_Q) \equiv JU'(J)\lambda_Q^{-2} + 2J^{-2/3}\left(w_1 + w_2\mathrm{I}_{\overline{\mathrm{C}}} - w_2\lambda_Q^2 J^{-2/3} - \tfrac{1}{3}(w_1\mathrm{I}_{\overline{\mathrm{C}}} + 2w_2\mathrm{II}_{\overline{\mathrm{C}}})\lambda_Q^{-2}J^{2/3}\right). \tag{3.24}$$

Eq. (3.22) represents a scalar non-linear equation in order to calculate λ_Q.

For the particular model (3.4) we have, using the kinematic relations $J = \lambda\lambda_Q^2$, $\mathrm{I}_{\overline{\mathrm{C}}} = J^{-2/3}(\lambda^2 + 2\lambda_Q^2)$ and $\mathrm{II}_{\overline{\mathrm{C}}} = J^{2/3}(\lambda^{-2} + 2\lambda_Q^{-2})$, the derivatives

$$U'(J) = \frac{K}{10}(J^4 - J^{-6}), \quad w_1 = c_{10} + 3\alpha\mathrm{I}_{\overline{\mathrm{C}}}^2 \text{ and } w_2 = c_{01}\frac{3}{2}\mathrm{II}_{\overline{\mathrm{C}}}^{1/2}. \tag{3.25}$$

Although the material parameters are developed for near-incompressibility, we investigate the variation of the compression modulus. In Fig. 4 the compression modulus K is varied and we compare the lateral stretch with the incompressible case. In the proximity to the undeformed state, the lateral stretch λ_Q is similar to the incompressible case (see Fig. 4). For a higher axial compression, $\lambda\to 0$, the lateral stretch increases monotonically, which one would expect in view of physical experiences or, equivalently, for a highly stretched specimen, $\lambda\to\infty$, λ_Q decreases and $\lambda_Q > 0$ holds. This fact differs from the results of Ehlers and Eipper (1998). On the basis of a few number of models of type (2.8) they emphasize the problem using strain-energy functions of type (3.1) which might show a non-physical behaviour in lateral direction. Of course, Fig. 4 does not prove that in such a diagram the slope is monotonically increasing or decreasing in the compression or tensile range respectively, but we do not observe for the small compression moduli

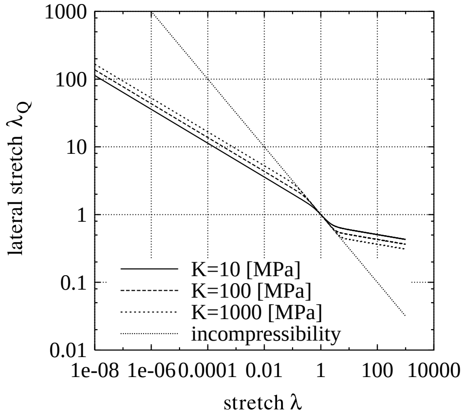

Fig. 4. Lateral stretch versus stretch diagram.

chosen the non-physical behaviour described. The behaviour in the lateral direction depends essentially on the interplay between the strain-energy parts $U(J)$ and $w(\mathrm{I}_{\overline{C}}, \mathrm{II}_{\overline{C}})$, which is very difficult to study due to the non-linear equation (3.22) and the difficulty to carry out experiments activating only $U(J)$ or $w(\mathrm{I}_{\overline{C}}, \mathrm{II}_{\overline{C}})$.

4. Conclusions

In this article we propose a new class of isotropic hyperelasticity relations for near-incompressibility based on principal invariants. We have proven the existence of a solution based on polyconvexity and coerciveness. Furthermore, we have shown that the extension of the strain energy of Arruda and Boyce (1993) containing unimodular tensorial quantities satisfies polyconvexity. Moreover, the extension of the generalized polynomial-type hyperelasticity is usually non-polyconvex. However, a particular dependence of the first and second invariant yields a polyconvex structure. The proposed strain-energy function for near-incompressibility has the specific advantage of being linear in the material parameters which, in the case of their identification, leads to a linear least-square problem with non-negative solution. Furthermore, the identification seems to be mostly non-sensitive.

Some studies of a particular model do not show a non-physical behaviour in the investigated examples, for instance, for the lateral expansion or stretch in uniaxial compression and tension, which results from the proposed volumetric part of the strain-energy function.

Appendix A. Necessary mathematical relationships

In the main part of the article some relationships are necessary. Here, we show the essential ones. First, for the subsequent proofs we need the Cayley-Hamilton theorem resulting from the characteristic polynomial for 3×3 matrices, $A \in \mathbb{M}^{3\times 3}$

$$\det(A - \lambda \mathbb{1}) = -\lambda^3 + (\operatorname{tr} A)\lambda^2 - (\operatorname{tr}\operatorname{adj} A)\lambda + \det A = 0$$

and which reads

$$-A^3 + (\operatorname{tr} A)A^2 - (\operatorname{tr}\operatorname{adj} A)A + (\det A)\mathbb{1} = 0, \tag{A.1}$$

$\mathbb{1}$ is the identity matrix. On the basis of these characteristics there are some relations between invariants and the eigenvalues of the matrices:

Lemma A.1 (Invariants). *For all real diagonalizable* $A \in \mathbb{M}^{3\times 3}$ *we set*

$$\mathrm{I}_A := \mathrm{tr}\,(A) = \lambda_1 + \lambda_2 + \lambda_3,$$
$$\mathrm{II}_A := \mathrm{tr}\,(\mathrm{adj}A) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3,$$
$$\mathrm{III}_A := \det A = \lambda_1\lambda_2\lambda_3.$$

Because of (A.1) *this implies*

$$(\mathrm{tr}\,F)^2 = \mathrm{tr}\,(F^2) + 2\mathrm{tr}\,(\mathrm{adj}F),$$
$$(\lambda_1 + \lambda_2 + \lambda_3)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3).$$

Lemma A.2 (Coefficients of the characteristic polynomial). *Let A be real diagonalizable and assume that* $\det A \geqslant 0$. *Then we have*

$$\mathrm{I}_A^2 \geqslant 3\mathrm{II}_A,$$
$$\mathrm{II}_A^2 \geqslant 3\mathrm{I}_A\mathrm{III}_A.$$

Proof. The second binomial expression shows that $\lambda_i\lambda_j \leqslant (1/2)\lambda_i^2 + (1/2)\lambda_j^2$ holds. Therefore $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \geqslant \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$. Hence

$$(\lambda_1 + \lambda_2 + \lambda_3)^2 = \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) + 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3) \geqslant 3(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)$$

which proves $\mathrm{I}_A^2 \geqslant 3\mathrm{II}_A$. In order to prove the second statement note that we may assume $\lambda_i(A) \neq 0$ without loss of generality since the statement is otherwise true anyway. Let therefore $\det A \neq 0$. Then the inverse $A^{-1} \in \mathbb{M}^{3\times 3}$ exists and with the first statement we know $\mathrm{I}_{A^{-1}}^2 \geqslant 3\mathrm{II}_{A^{-1}}^2$. Moreover $\hat{\lambda}_i(A^{-1}) = (1/\lambda_i(A))$. Therefore

$$\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right)^2 \geqslant 3\left(\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_3\lambda_1}\right),$$
$$\left(\frac{\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3}{\lambda_1\lambda_2\lambda_3}\right)^2 \geqslant 3\left(\frac{\lambda_1 + \lambda_2 + \lambda_3}{\lambda_1\lambda_2\lambda_3}\right),$$
$$(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)^2 \geqslant 3(\lambda_1 + \lambda_2 + \lambda_3)\cdot(\lambda_1\lambda_2\lambda_3)$$

which is just $\mathrm{II}_A^2 \geqslant 3\mathrm{I}_A\mathrm{III}_A$. $\square$

In the proof of Lemma 2.2 we need some relationships between different invariants. These are shown in the following:

Corollary A.3 (Estimates between $\|F\|$, $\|\mathrm{adj}F\|$ and $\det F$). *Let* $F \in \mathbb{M}^{3\times 3}$. *Then we have*

$$\|F\|^3 \geqslant 3\sqrt{3}\det F,$$
$$\|F\|^2 \geqslant \sqrt{3}\|\mathrm{adj}F\|,$$
$$\|\mathrm{adj}F\|^3 \geqslant 3\sqrt{3}(\det F)^2,$$
$$\|F\|^2 = \langle F^{\mathrm{T}}F, \mathbb{1}\rangle \leqslant \sqrt{3}\|F^{\mathrm{T}}F\|.$$

Proof. Set $C = F^{\mathrm{T}}F$ (right Cauchy–Green tensor). The symmetry of C ensures the applicability of the preceding Lemma A.2. Thus

$$\mathrm{I}_A = \mathrm{tr}\,(F^{\mathrm{T}}F) = \|F\|^2,$$
$$\mathrm{II}_A = \mathrm{tr}\,(\mathrm{adj}\,(F^{\mathrm{T}}F)) = \mathrm{tr}\,(\mathrm{adj}\,F\,\mathrm{adj}\,F^{\mathrm{T}}) = \|\mathrm{adj}\,F\|^2,$$
$$\mathrm{III}_A = \det(F^{\mathrm{T}}F) = (\det F)^2$$

and also

$$\mathrm{I}_C^2 \geqslant 3\mathrm{II}_C \iff \|F\|^2 \geqslant \sqrt{3}\|\mathrm{adj}\,F\|,$$
$$\mathrm{II}_C^2 \geqslant 3\mathrm{I}_C\mathrm{III}_C \iff \|\mathrm{adj}\,F\|^2 \geqslant \sqrt{3}\|F\|\det F.$$

The last two lines lead directly to the second statement. The last statement is only a simple algebraic estimate. □

Appendix B. Convexity

In order to understand polyconvexity, we start with some properties of convexity. In the following, one can imagine that W symbolizes the strain-energy function, F the deformation gradient and C the matrix representation of the right Cauchy–Green tensor.

Definition B.1 (*Convexity*). Let K be a convex set and let $W : K \to \mathbb{R}$. We say that W is convex if

$$W(\lambda F_1 + (1-\lambda)F_2) \leqslant \lambda W(F_1) + (1-\lambda)W(F_2)$$

for all $F_1, F_2 \in K$ and $\lambda \in (0,1)$.

Note that in this definition it is necessary that the function W be defined on a convex set K.

Lemma B.2 (Second derivative and convexity). *Let K be a convex set and let $W : K \to \mathbb{R}$ be twice differentiable. Then the following statements are equivalent:*

1. *W is convex,*
2. *$D^2W(F).(H,H) \geqslant 0 \quad \forall F \in K, \forall H \in Lin(K)$,*

where $Lin(K)$ is the linear hull of K.

Proof. See Rockafellar (1970, p. 27). □

Remark B.3. In order that $W : K \to \mathbb{R}$ is convex, it is not sufficient to assume only

$$D^2W(F).(H,H) \geqslant 0$$

for all $F \in K, \forall H \in K$. Since, for example, with $W(C) = \det C$, $W : \mathbb{S}\mathrm{ym}^+ \to \mathbb{R}$, we have $K = \mathbb{S}\mathrm{ym}^+$ is a convex set (cone) and

$$D^2W(C).(H,H) = 2\langle C, \mathrm{adj}\,H\rangle \geqslant 0$$

for $C, H \in \mathbb{S}\mathrm{ym}^+$, but $W(C) = \det C$ is not convex.

If a function is given on $C = F^{\mathrm{T}}F$ it is possible to relate convexity properties of a function defined on C and the corresponding function defined on F.

Lemma B.4 (Convexity on $\mathbb{M}^{3\times3}$ and $\mathbb{S}\mathrm{ym}^+(3)$). *Let $C \in \mathbb{S}\mathrm{ym}^+(3)$ and $\Psi : \mathbb{S}\mathrm{ym}^+(3) \to \mathbb{R}$ and assume that for all $H \in \mathbb{S}\mathrm{ym}(3)$: $(D_C^2\Psi(C)\cdot H, H) \geqslant 0$ and $D_C\Psi(C) \in \mathbb{S}\mathrm{ym}_0^+(3)$. Then the function*

$$W : \mathbb{M}^{3\times3} \to \mathbb{R}, \quad F \mapsto \Psi(F^{\mathrm{T}}F) \tag{B.1}$$

is convex.

Proof. See Neff (2000). □

Lemma B.5 (Convexity of the square). *Let $P : \mathbb{R}^n \to \mathbb{R}$ be convex and $P(Z) \geqslant 0$. Then the function $Z \in \mathbb{R}^n \mapsto E(Z) = P(Z)\cdot P(Z)$ is convex.*

Proof. First assume that P is a smooth function. The second differential of $E(Z) = P(Z)\cdot P(Z)$ is easy to calculate. We get

$$DE(Z).H = P(Z)\cdot DP(Z).H + DP(Z).H\cdot P(Z),$$
$$D^2E(Z).(H,H) = 2\big(P(Z)\cdot D^2P(Z).(H,H) + DP(Z).H\cdot DP(Z).H\big) \geqslant 0.$$

Hence $E(Z)$ is convex. In the non-smooth case we proceed as follows:

$$E(\lambda Z_1 + (1-\lambda)Z_2) = P(\lambda Z_1 + (1-\lambda)Z_2)\cdot P(\lambda Z_1 + (1-\lambda)Z_2).$$

The assumed convexity of P shows that

$$P(\lambda Z_1 + (1-\lambda)Z_2) \leqslant \lambda P(Z_1) + (1-\lambda)P(Z_2).$$

Since the square function is a monotonically increasing function for positive values and by assumption $\lambda P(Z_1) + (1-\lambda)P(Z_2)$ is positive, we obtain the estimate

$$E(\lambda Z_1 + (1-\lambda)Z_2) \leqslant (\lambda P(Z_1) + (1-\lambda)P(Z_2))^2.$$

However, since the square function is itself convex, we may proceed to write

$$E(\lambda Z_1 + (1-\lambda)Z_2) \leqslant \lambda P^2(Z_1) + (1-\lambda)P^2(Z_2) = \lambda E(Z_1) + (1-\lambda)E(Z_2).$$

The proof is complete. □

Corollary B.6. *Let $P : \mathbb{R}^n \to \mathbb{R}$ be convex and assume that $P(Z) \geqslant 0$. Then the function*

$$Z \in \mathbb{R}^n \mapsto [P(Z)]^p, \quad p \geqslant 1$$

is convex.

Proof. The same ideas as before apply to this situation. □

Lemma B.7 (Convexity and monotone composition). *Let $P : \mathbb{R}^n \to \mathbb{R}$ be convex and let $m : \mathbb{R} \to \mathbb{R}$ be convex and monotone increasing. Then the function $\mathbb{R}^n \to \mathbb{R}$, $X \mapsto m(P(X))$ is convex.*

Proof. A direct check of the convexity condition. □

Since we have products of different positive scalar functions—see, for example, the strain-energy function (2.13)—it is useful to stress the non-convexity of mixed products in general. Let $P_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2$, be convex and assume $P_i \geqslant 0$. Then the functions

$$Z \in \mathbb{R}^n \mapsto P_1(Z) \cdot P_2(Z)$$
$$Z \in \mathbb{R}^n \mapsto P_1^p(Z) \cdot P_2^p(Z), \quad p \geqslant 1$$

are in general non-convex. $x \to x^2(x-1)^2$ and $x \to e^x x^2$ may serve as simple examples. The function $(x, y) \to x^2 y^2$ may serve as an example where functions in different variables are convex and positive, but their product is not convex.

Appendix C. Polyconvexity

In respect of the aforementioned observations of convexity, we define in this subsection *polyconvexity* and relate it to *ellipticity* and *quasi-convexity*.

Definition C.1 (*Polyconvexity*). Let $W \in C^2(\mathbb{M}^{3\times 3}, \mathbb{R})$ be a given scalar-valued energy density. We say that $F \mapsto W(F)$ is polyconvex if and only if there exists a function $P : \mathbb{M}^{3\times 3} \times \mathbb{M}^{3\times 3} \times \mathbb{R} \to \mathbb{R}$ (in general non-unique) such that

$$W(F) = P(F, \operatorname{adj} F, \det F)$$

and the function $\mathbb{R}^{19} \to \mathbb{R}, (X, Y, Z) \mapsto P(X, Y, Z)$ is convex.

A consequence of this definition for a more restrictive class of energy densities is

Corollary C.2 (Additive polyconvex functions). *Let* $W(F) = W_1(F) + W_2(\operatorname{adj} F) + W_3(\det F)$. *If the* W_i, $i = 1, \ldots, 3$, *are convex with respect to their arguments, then* W *is altogether polyconvex.*

This corollary will be one of our main tools in constructing polyconvex strain energies: as we have seen before in the main part of the article, we identify functions which are convex on $\mathbb{M}^{3\times 3}$ and $\mathbb{R}$ and then take positive combinations of them. Let us now relate polyconvexity to quasi-convexity and ellipticity.

Definition C.3 (*Quasi-convexity*). We say that the elastic free energy W is quasi-convex whenever for arbitrary $\Omega \subset \mathbb{R}^3$ and all $F \in \mathbb{M}^{3\times 3}$ and all $v \in C_0^\infty(\Omega)$ we have

$$W(F) \cdot |\Omega| = \int_\Omega W(F)\,\mathrm{d}X \leqslant \int_\Omega W(F + \nabla v(X))\,\mathrm{d}X.$$

This means that the homogeneous solution $\nabla\phi = F$ of the homogeneous boundary-value problem

$$\operatorname{Div} D_F W(\nabla\phi) = 0,$$
$$\phi_{\partial\Omega}(X) = F.X + c$$

is automatically a global minimizer. It is clear that this condition is a non-local stability condition which is difficult to handle. Every quasi-convex function is automatically elliptic.

Definition C.4 (*Ellipticity*). We say that

(1) Elastic free energy $W \in C^2(\mathbb{M}^{3\times 3}, \mathbb{R})$ leads to a uniformly elliptic equilibrium system whenever the so-called uniform Legendre-Hadamard condition

$$\exists c^+ > 0 \forall F \in \mathbb{M}^{3\times 3} : \forall \xi, \eta \in \mathbb{R}^3 : \quad D_F^2 W(F).(\xi \otimes \eta, \xi \otimes \eta) \geqslant c^+ \cdot \|\xi\|^2 \|\eta\|^2$$

holds.

(2) W is strictly elliptic if and only if the strict Legendre-Hadamard condition

$$\forall F \in \mathbb{M}^{3\times 3} : \forall \xi, \eta \in \mathbb{R}^3 : \quad D_F^2 W(F).(\xi \otimes \eta, \xi \otimes \eta) > 0$$

holds.

(3) Elastic free energy W is strictly rank-one convex if the function $f : \mathbb{R} \to \mathbb{R}, f(t) = W(F + t \cdot (\xi \otimes \eta))$ is strictly convex for all $F \in \mathbb{M}^{3\times 3}$ and all $\xi, \eta \in \mathbb{R}^3$.

Theorem C.5 (Rank one convexity and eigenvalues). *Let $W(F) = \Phi(\lambda_1, \lambda_2, \lambda_3)$, where Φ is symmetric and the λ_i's are the eigenvalues of the matrix representation of the right stretch tensor $U = (F^{\mathrm{T}}F)^{1/2}$. If W is rank one convex and $\Phi \in C^2(\mathbb{R}^3)$ then*

$$\frac{\partial^2 \Phi}{\partial \lambda_i^2}(\lambda_1, \lambda_2, \lambda_3) \geqslant 0.$$

Proof. This is Proposition 1.2 of Dacorogna (1989, p. 254). □

The decisive property in the context treated here is the following well known result.

Theorem C.6 (Polyconvexity implies ellipticity). *Let the stored energy W be sufficiently smooth. Then, if W is polyconvex, it is quasi-convex and elliptic. Moreover rank-one ellipticity and ellipticity are equivalent. Let W be strictly elliptic. Then the Baker–Ericksen inequalities (see* Baker and Ericksen, 1954*;* Marsden and Hughes, 1983*) are satisfied.*

Proof. Standard result in the calculus of variations (see Dacorogna, 1989). We note that the reverse implications are in general not true. □

Although we restrict our applications to purely isochoric terms, we study some more general expressions:

Lemma C.7 (Generic polyconvex terms). *Let $F \in \mathbb{M}^{3\times 3}$. Then each of the following terms is polyconvex*:

(1) $$\frac{(\mathrm{tr}\,(F^{\mathrm{T}}F))^k}{\det(F^{\mathrm{T}}F)^{1/3}}, \quad k \geqslant 1.$$

(2) $$\frac{(\mathrm{tr}\,(\mathrm{adj}\,(F^{\mathrm{T}}F)))^k}{\det(F^{\mathrm{T}}F)^{1/3}}, \quad k \geqslant 1.$$

Proof. (1) We consider the term

$$\frac{(\mathrm{tr}\,(F^{\mathrm{T}}F))^k}{\det(F^{\mathrm{T}}F)^{1/3}} = \frac{\|F\|^{2k}}{(\det F)^{2/3}}.$$

We have already shown (see Section 2.3.1) that the function $P(x, y) = (1/x^\alpha) \cdot y^p$ is convex provided that $\alpha = 2/3$ and $p = 2k \geqslant 2$. Now define a new function

$$\widehat{W}(F,\widetilde{J}) := P(\widetilde{J},\|F\|) = \frac{\|F\|^{2k}}{\widetilde{J}^{2/3}}.$$

Note that through the monotonicity of the 2kth power for positive arguments we have the inequality

$$\|\lambda F_1 + (1-\lambda)F_2\|^{2k} \leqslant (\lambda\|F_1\| + (1-\lambda)\|F_2\|)^{2k}. \tag{C.1}$$

It remains to check the convexity of $\widehat{W}(F,\widetilde{J})$. To this end

$$\widehat{W}(\lambda F_1 + (1-\lambda)F_2, \lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2) = P(\lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2, \|\lambda F_1 + (1-\lambda)F_2\|) = \frac{\|\lambda F_1 + (1-\lambda)F_2\|^{2k}}{(\lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2)^{2/3}}.$$

With (C.1) we have

$$\widehat{W}(\lambda F_1 + (1-\lambda)F_2, \lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2) \leqslant \frac{(\lambda\|F_1\| + (1-\lambda)\|F_2\|)^{2k}}{(\lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2)^{2/3}}$$
$$= P(\lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2, \lambda\|F_1\| + (1-\lambda)\|F_2\|).$$

The convexity of P yields

$$\widehat{W}(\lambda F_1 + (1-\lambda)F_2, \lambda\widetilde{J}_1 + (1-\lambda)\widetilde{J}_2) \leqslant \lambda P(\widetilde{J}, \|F_1\|) + (1-\lambda)P(\widetilde{J}_2, \|F_2\|)$$
$$= \lambda\widehat{W}(F_1,\widetilde{J}_1) + (1-\lambda)\widehat{W}(F_2,\widetilde{J}_2).$$

The proof is completed in terms of the correct extension (2.16).
(2) Set

$$\frac{\operatorname{tr}(\operatorname{adj}(F^{\mathrm{T}}F))^k}{\det(F^{\mathrm{T}}F)^{1/3}} = \frac{\|\operatorname{adj}F\|^{2k}}{(\det F)^{2/3}}$$

and we proceed as in case 1. □

Other polyconvex functions are, for example, the following terms, which we call *Generic exponential polyconvex terms*. For a given deformation gradient $F \in \mathbb{M}^{3\times 3}$ each of the following terms is polyconvex:

(1) $\exp\left[\dfrac{\operatorname{tr}(F^{\mathrm{T}}F)^k}{\det(F^{\mathrm{T}}F)^{1/3}}\right], \quad k \geqslant 1.$

(2) $\exp\left[\dfrac{\operatorname{tr}(\operatorname{adj}(F^{\mathrm{T}}F))^k}{\det(F^{\mathrm{T}}F)^{1/3}}\right], \quad k \geqslant 1.$

(3) $\exp(W(F))$ if $W(F)$ is polyconvex.

In view of the preceding Lemma each argument of the exponential is polyconvex. Since exp is convex and monotonically increasing it preserves the underlying convexity. Hence the composition is polyconvex. Note, however, that these functions alone are not stress-free in the reference configuration, i.e. we have to combine them with other polyconvex functions in such a way that zero stresses are satisfied due to an undeformed state.

Appendix D. Coercivity

In order to propose a class of strain-energy functions satisfying an existence theorem, we start with

Definition D.1 (*Coercivity*). Let $I(\phi)$ be the elastic stored energy functional with the deformation $\phi(X,t)$. We say that I is q-**coercive** whenever

$$I(\phi) \leqslant K \Rightarrow \|\phi\|_{1,q,\Omega} \leqslant \widetilde{K},$$

where $\|\phi\|_{1,q,\Omega}$ is the Sobolev norm on the space $W^{1,q}(\Omega)$.

In view of Section 3 we investigate the coercivity of a special energy which contains those terms already proved to be polyconvex, see Corollary 2.3, and which yields a stress-free initial configuration in a natural way.

Lemma D.2 (Coercivity of special energy). *For $J = \det F$ and $\overline{C} = C/(\det C)^{1/3}$ with $C = F^{\mathrm{T}}F$ let the elastic stored energy density be given by*

$$W(F) = K\left((\det F)^5 + \frac{1}{(\det F)^5} - 2\right) + \alpha\left(\left(\frac{\|F\|^2}{(\det F)^{2/3}}\right)^3 - 3^3\right) + \sum_{i=1}^{m} c_{i0}\left(\frac{\|F\|^2}{(\det F)^{2/3}} - 3\right)^i + \sum_{j=1}^{n} c_{0j}\left(\frac{\|\operatorname{adj} F\|^3}{(\det F)^2} - 3\sqrt{3}\right)^j \tag{D.1}$$

$$= K\left(J^5 + \frac{1}{J^5} - 2\right) + \alpha((\operatorname{tr}\overline{C})^3 - 3^3) + \sum_{i=1}^{m} c_{i0}(\operatorname{tr}\overline{C} - 3)^i + \sum_{j=1}^{n} c_{0j}\left((\operatorname{tr}\operatorname{adj}\overline{C})^{3/2} - 3\sqrt{3}\right)^j \tag{D.2}$$

with $K, \alpha > 0$, $c_{i0} \geqslant 0$, $i = 1, \dots, m$, and $c_{0j} \geqslant 0$, $j = 1, \dots, n$. Then

$$I(\phi) = \int_\Omega W(\nabla\phi)\,\mathrm{d}X$$

is coercive for $q = 4$.

Proof.

$$\begin{aligned}
\|F\|^q_{q,\Omega} &= \left\| \frac{F}{(\det F)^{1/3}} (\det F)^{1/3} \right\|^q_{q,\Omega} \\
&= \int_\Omega \left\| \frac{F}{(\det F)^{1/3}} \right\|^q |\det F|^{q/3}\,\mathrm{d}X \quad \text{apply Youngs inequality, } xy \leqslant \frac{1}{a}x^a + \frac{1}{b}y^b, \text{ with } \frac{1}{a} + \frac{1}{b} = 1 \\
&\leqslant \int_\Omega \left(\frac{1}{a} \left\| \frac{F}{(\det F)^{1/3}} \right\|^{qa} + \frac{1}{b} |\det F|^{qb/3} \right) \mathrm{d}X \quad \text{taking } a = \frac{3}{2},\ b = 3 \text{ yields} \\
&= \int_\Omega \left(\frac{2}{3} \left\| \frac{F}{(\det F)^{1/3}} \right\|^{3q/2} + \frac{1}{3} |\det F|^q \right) \mathrm{d}X \quad \text{for } q = 4 \text{ this shows} \\
&= \int_\Omega \left(\frac{2}{3} \left\| \frac{F}{(\det F)^{1/3}} \right\|^6 + \frac{1}{3} |\det C|^2 \right) \mathrm{d}X
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_{\Omega}\left(\frac{2}{3}\left(\frac{\|F\|^2}{(\det F)^{2/3}}\right)^3+\frac{1}{3}(1+|\det F|^5)\right)\mathrm{d}X \\
&\leqslant \frac{2}{3\alpha}I(\phi)+3^3+\frac{1}{3K}I(\phi)+2+\frac{1}{3}|\Omega| \\
&\leqslant \left(\frac{2}{3\alpha}+\frac{1}{3K}\right)I(\phi)+2+3^3+\frac{1}{3}|\Omega|.
\end{aligned}
$$

Applying Poincaré's inequality will complete the proof if Dirichlet boundary conditions are applied. □

The strain energy of type (D.1) or (D.2) respectively, contains the term $\alpha(\cdots)$, $\alpha > 0$, which is necessary to guarantee the coercivity in conjunction with the chosen volumetric term, $K(\cdots)$, $K > 0$. Having proved the coercivity and polyconvexity of the polynomial ansatz chosen, it is a standard matter to prove the existence of a minimizer.

Theorem D.3 (Existence of minimizers). *Let the reference configuration $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain and let $\partial\Omega_1$ be a part of the boundary $\partial\Omega$ with non-vanishing Lebesgue measure. Assume that $I(\phi) = \int_\Omega W(\nabla\phi(X))\,\mathrm{d}X$ with W as in (D.1). Let $\phi_0 \in W^{1,4}(\Omega)$ be given with $I(\phi_0) < \infty$. Then the problem*

$$
\inf\left\{I(\phi)=\int_{\Omega}W(\nabla\phi(X))\,\mathrm{d}X,\ \phi(X)=\phi_0(X)\ \text{for } X\in\partial\Omega_1,\ \phi\in W^{1,4}(\Omega)\right\}
$$

admits at least one solution. Formally, this solution corresponds to a solution of the boundary-value problem

$$
\begin{aligned}
&\mathrm{Div}\, D_F W(\nabla\phi)=0,\\
&\phi(X)=\phi_0(X),\quad X\in\partial\Omega_1.
\end{aligned}
$$

Proof. It has been shown in Corollary 2.3 and Lemma D.2 that the energy (D.1) or (D.2) is polyconvex and coercive on $W^{1,4}(\Omega)$. Since $I(\phi) \geqslant 0$ and $I(\phi_0) < \infty$ the infimum exists and the direct methods of variation, together with weak convergence, yield the existence of at least one minimizer. □

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